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A COMPARISON OF CONVERGENCE ACCELERATION SCHEMES FOR EIGENFUNCTION EXPANSIONS OF PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

Different possibilities for the enhancement of convergence rates in eigenfunction expansions are investigated in the realm of integral transform solutions for partial differential equations. A representative parabolic problem is chosen to illustrate two schemes and their combinations; a filtering technique and an integral balance approach. Numerical results are presented to confirm the relative merits in each proposed procedure.

KEY WORDS Convergence rates Eigenfunction expansions Partial differential equations

NOMENCLATURE

A = Cross-sectional area
A_r = A reference cross-sectional area
b = Decay constant for the exponential functional form adopted as the fin's base temperature
h = Heat transfer coefficient between the lateral surface and the fluid
h_r = A reference heat transfer coefficient
k = Thermal conductivity of the fin material
K = Dimensionless area as defined by Equation (2b)
L_r = A reference length
M = Dimensionless parameter defined by Equation (2e)
p_r = Perimeter of the reference cross-sectional area A_r
S = Lateral area of the fin
t = Time
T = Temperature of the fin
T_r = A reference temperature
T_∞ = Temperature of the surrounding fluid
x = Position
X = Dimensionless position as defined by Equation (2a)
X_l = Dimensionless length of the fin
W = Dimensionless parameter defined by Equation (2g)

Greek symbols
α = diffusivity
θ = dimensionless temperature as defined by Equation (2d)
τ = dimensionless time as defined by Equation (2c)

INTRODUCTION

The Integral Transform Method, in its classical sense[1,2], is a well-known analytical approach for the solution of certain classes of linear diffusion problems, based on expansions of the original potentials in terms of an associated eigenvalue problem. A major aspect in the practical implementation of such methodology is the eventual need for improving the convergence behaviour of the resulting eigenfunction expansions. Within the context of those classes of problems that may be handled exactly, a number of convergence acceleration schemes were proposed over the last few years[1-4], essentially originated from the splitting-up of the original

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partial differential system into simpler problems. However, the expressions "so developed" are limited to time-dependent source functions represented by exponentials and q-order polynomials, and the analytical involvement makes it not so practical for more general situations of arbitrary source functions and multidimensional and/or non-linear problems.

Motivated by the recent developments in the so-called Generalized Integral Transform Technique (GITT), reviewed in references 5 to 7, for the hybrid numerical-analytical solution of non-linear diffusion and convection-diffusion problems, an alternative approach based on integral balances was proposed[5-9], and different filtering schemes were employed throughout the sparse literature on this methodology[5,7,10,11]. The aim was to provide simpler convergence enhancement procedures, in order to maintain the applicability of the formal solution approach into the widest possible range of posed problems in heat and fluid flow, and to within a mild degree of analytical involvement, for compatibility with the development of automatic solvers for partial differential equations in such fields.

The present work brings a systematic description and critical comparison of such alternative approaches, enriched by their combinations as one single scheme. A typical heat diffusion problem; the transient temperature profile in a fin, is selected to illustrate the different possibilities of applying the improved eigenfunction expansions, and numerical results are presented to investigate the relative merits in each proposed scheme.

PROBLEM FORMULATION

A representative heat conduction problem is selected for illustration purposes, related to the determination of transient temperature distributions along fins of constant profile.

The generalized transient fin equation can be stated, in dimensionless form, as[5]

$$K(X)\frac{\partial\theta(X,\tau)}{\partial\tau}=\frac{\partial}{\partial X}\left[K(X)\frac{\partial\theta(X,\tau)}{\partial X}\right]-M^2W(X,\tau)\,\theta\,(X,\tau),0<X<X_t,\tau>0 \tag{1}$$

where the basic assumptions behind such formulation include the usual hypothesis of a uniform temperature in any cross-section (one dimensional problem), homogeneous material, constant thermal conductivity, no energy sources or sinks within the fin, and uniform temperature of the surrounding fluid[2].

The various dimensionless groups are defined as[5]:

$$X=\frac{x}{L_r};K(X)=\frac{A(x)}{A_r};\tau=\frac{\alpha t}{L_r^2};\theta(X,\tau)=\frac{T(x,t)-T_\infty}{T_r-T_\infty};$$

$$M^2=\frac{p_rh_r}{kA_r}L_r^2;W(X,\tau)=\frac{h(x,t)}{p_rhr}\frac{dS(x)}{dx}. \tag{2a-f}$$

For a rectangular shaped fin and constant heat transfer coefficient ($h(x,t)=h_r$), $K(X)=W(X,\tau)=1$ as shown in Reference 2, reducing equation (1) to

$$\frac{\partial\theta}{\partial\tau}=\frac{\partial^2\theta}{\partial X^2}-M^2\theta,0<X<1,\tau>0 \tag{3a}$$

and the adopted initial and boundary conditions are given, respectively, by

$$\theta(X,0)=\theta_p(X) \tag{3b}$$

$$\theta(0,\tau)=\theta_b(\tau) \tag{3c}$$

$$\frac{\partial\theta(1,\tau)}{\partial X}=0 \tag{3d}$$

neglecting heat transfer at the fin's tip for the present case.

As an initial condition, equation (3b), the steady-state temperature profile for constant temperature at the fin's base was adopted, i.e., $\theta_p(X)$ is the solution of the following problem:

$$\frac{d^2\theta_p}{dX^2} - M^2\theta_p = 0, 0 < X < 1 \tag{4a}$$

$$\theta_p(0) = 1 \tag{4b}$$

$$\frac{d\theta_p(1)}{dX} = 0 \tag{4c}$$

which furnishes

$$\theta_p(X) = \cosh(MX) - \tanh(M)\sinh(MX). \tag{5}$$

The functional form for the time varying fin's base temperature was chosen as

$$\theta_b(\tau) = e^{-b\tau}, b > 0 \tag{6}$$

representing, for instance, the base's temperature decay following a shut-down operation of the associated heat transfer equipment.

FORMAL SOLUTION

Following the formalism in the classical integral transform technique[1,2,5], the associated eigenvalue problem is obtained as

$$\frac{d^2\psi_i}{dX^2} + (\mu_i^2 - M^2)\psi_i = 0, 0 < X < 1 \tag{7a}$$

$$\psi_i(0) = 0 \tag{7b}$$

$$\frac{d\psi_i(1)}{dX} = 0 \tag{7c}$$

yielding,

$$\text{eigenfunctions:} \quad \psi_i(X) = \sin(\sqrt{\mu_i^2 - M^2}X) \tag{8a}$$

$$\text{eigenvalues:} \quad \mu_i = \sqrt{\left[(2i-1)\frac{\pi}{2}\right]^2 + M^2}, i = 1, 2, 3, \ldots \tag{8b}$$

$$\text{and norms:} \quad \mathrm{N_i} = 1/2, i = 1, 2, 3, \ldots \tag{8c}$$

This auxiliary problem allows for the definition of the integral transform pair:

$$\bar{\theta}_i(\tau) = \int_0^1 \psi_i(X)\,\theta(X,\tau)\,dX \qquad \text{Transform} \tag{9a}$$

$$\theta(X,\tau) = \sum_{i=1}^{\infty} \frac{1}{N_i}\psi_i(X)\,\bar{\theta}_i(\tau) \qquad \text{Inversion.} \tag{9b}$$

The remaining analysis is well documented in references 1, 2 and 5 and should not be repeated here. The basic idea consists of transforming the original partial differential equation system (equation (3)) into an infinite system of decoupled ordinary differential equations, i.e.,

$$\frac{d\bar{\theta}_i(\tau)}{d\tau} + \mu_i^2\bar{\theta}_i(\tau) = g_i(\tau) \tag{10a}$$

$$\bar{\theta}_i(0) = \bar{f}_i \tag{10b}$$

which is readily solved to furnish

$$\bar{\theta}_i(\tau) = e^{-\mu_i^2\tau}\left[\bar{f}_i + \int_0^\tau g_i(\tau')\, e^{\mu_i^2\tau'} d\tau'\right] \quad (10c)$$

Then, the formal solution is directly obtained from the equation 9b as

$$\theta(X,\tau) = \sum_{i=1}^{\infty} \frac{1}{N_i} e^{-\mu_i^2\tau}\psi_i(X)\left[\bar{f}_i + \int_0^\tau g_i(\tau')\, e^{\mu_i^2\tau'} d\tau'\right] \quad (11a)$$

where the transformed initial condition, $\bar{f}_i$, is given by

$$\bar{f}_i = \int_0^1 \theta_p(X)\,\psi_i(X)\,dX = \frac{1}{\mu_i^2}\sqrt{\mu_i^2 - M^2} \quad (11b)$$

and g_i, originating from the non-homogeneity at the boundaries, is obtained as

$$g_i(\tau') = \left[\sqrt{\mu_i^2 - M^2}\right]\theta_b(\tau'). \quad (11c)$$

Substituting equations (11b) and (11c) into equation (11a) and evaluating the integral, one obtains for the adopted form of θ_b (equation 6) the following working expression

$$\theta(X,\tau) = 2\sum_{i=1}^{\infty} \frac{\sqrt{\mu_i^2 - M^2}}{\mu_i^2 - b}\left[e^{-b\tau} - \frac{b}{\mu_i^2} e^{-\mu_i^2\tau}\right]\sin\left(\sqrt{\mu_i^2 - M^2}\,X\right). \quad (12)$$

As can be seen, the solution, equation (12) does not reproduce the boundary condition at $X = 0$, since equation (7b) is not compatible with equation (3c). This fact markedly affects the convergence behaviour of the formal solution, particularly at points approaching $X = 0$, and for any practical purpose this expression, (equation (12)) should be avoided, and convergence enhancement schemes must be invoked.

FILTERING SCHEME

The filtering approach to enhance the convergence of eigenfunction expansions is based on the idea of eliminating, or at least reducing, the influence of the problem source terms (both in the equation itself and the associated boundary conditions). Such source terms act in the sense of deviating the resulting formal solution from the well-behaved exponentially decaying expressions for a fully homogeneous problem, bringing up undesirable characteristics for the series convergence behaviour. Therefore, extracting simpler formulations from the original problem that contain, at least in part, some information represented by these source terms, may yield improved solutions obtained from the remaining partial differential formulation that results from application of the filter, providing a net weakening effect on the source terms (or their complete remotion from the problem).

For the present application, the following filtering strategy is proposed:

$$\theta(X,\tau) = \theta_s(X\,|\,\tau) + \theta_t(X,\tau) \quad (13)$$

where the filtering solution, $\theta_s(X|\tau)$, is obtained from the following simpler problem (τ is now regarded as a parameter),

$$\frac{d^2\theta_s}{dX^2} - M^2\theta_s = 0 \quad (14a)$$

$$\theta_s(0\,|\,\tau) = \theta_b(\tau) \quad (14b)$$

$$\frac{d\theta_s(1\,|\,\tau)}{dX} = 0. \quad (14c)$$

From substitution of the proposed expression, equation (13), into the original system, equation (3), the new task is then to solve the following transient problem for $\theta_t(X,\tau)$:

$$\frac{\partial \theta_t}{\partial \tau} = \frac{\partial^2 \theta_t}{\partial X^2} - M^2 \theta_t - \frac{\partial \theta_s}{\partial \tau} \tag{15a}$$

$$\theta_t(X, 0) = \theta_p(X) - \theta_s(X|0) \tag{15b}$$

$$\theta_t(0, \tau) = 0 \tag{15c}$$

$$\frac{\partial \theta_t(1, \tau)}{\partial X} = 0. \tag{15d}$$

For this new problem, the boundary condition source term has been completely eliminated, while the equation source term has been at least filtered to some extent.

Problem (14) is similar to problem (4) and is easily solved as

$$\theta_s(X \mid \tau) = \theta_b(\tau)[\cosh(MX) - \tanh(M)\sinh(MX)]. \tag{16}$$

Again, problem (15) is solved through the classical integral transform technique. The eigenvalue problem is the same as in equation (3), generating the same eigenfunctions, eigenvalues and norms, equation (8a-c). The general solution, equation (11a), is again applied and the transformed initial condition, $\bar{f}_i$, is obtained from:

$$\bar{f}_i = \int_0^1 [\theta_p(X) - \theta_s(X/0)]\, \psi_i(X)\, dX \tag{17a}$$

and g_i, now the transformed filtered source term $(-\partial\theta_s/\partial\tau)$ in equation (15a), is evaluated as

$$g_i(\tau') = -\frac{\partial \theta_b(\tau')}{\partial \tau'} \frac{\sqrt{\mu_i^2 - M^2}}{\mu_i^2}. \tag{17b}$$

Then, formula (11a) and equation (13) provide the final solution from this filtering approach as:

$$\theta(X, \tau) = e^{-b\tau}[\cosh(MX) - \tanh(M)\sinh(MX)]$$

$$+ 2\sum_{i=1}^{\infty} \frac{\sqrt{\mu_i^2 - M^2}}{\mu_i^2 - b} \left[\frac{b}{\mu_i^2} e^{bt} - \frac{b}{\mu_i^2} e^{-\mu_i^2 \tau}\right] \sin\left(\sqrt{\mu_i^2 - M^2}\, X\right). \tag{18}$$

Thus, solution (18) now satisfies the boundary condition at $X = 0$ identically, and this expression is, therefore, expected to present improved convergence behaviour over the formal solution, equation (12). The filtering scheme may proceed in the direction of progressively decreasing the relative importance of the remaining source terms that could not be eliminated through the last filter employed.

Following this reasoning, one can propose a double filter to the present problem, by letting

$$\theta_t(X, \tau) = \theta_{ss}(X \mid \tau) + \theta_{tt}(X, \tau) \tag{19}$$

where $\theta_{ss}(X|\tau)$ is obtained from the simpler problem below, that incorporates the remaining equation source term,

$$\frac{d^2 \theta_{ss}}{dX^2} - M^2 \theta_{ss} - \frac{\partial \theta_s}{\partial \tau} = 0 \tag{20a}$$

$$\theta_{ss}(0 \mid \tau) = 0 \tag{20b}$$

$$\frac{d\theta_{ss}(1 \mid \tau)}{dX} = 0 \tag{20c}$$

and $\theta_{tt}(X,\tau)$ comes from the twice filtered partial differential system as follows:

$$\frac{\partial\theta_{tt}}{\partial\tau}=\frac{\partial^2\theta_{tt}}{\partial X^2}-M^2\theta_{tt}-\frac{\partial\theta_{ss}}{\partial\tau} \tag{21a}$$

$$\theta_{tt}(X,0)=\theta_p(X)-\theta_s(X\,|\,0)-\theta_{ss}(X\,|\,0) \tag{21b}$$

$$\theta_{tt}(0,\tau)=0 \tag{21c}$$

$$\frac{\partial\theta_{tt}(1,\tau)}{\partial X}=0 \tag{21d}$$

where the solution of problem (20) yields:

$$\theta_{ss}(X\,|\,\tau)=\frac{1}{2M}\frac{d\theta_b}{d\tau}\left\{-\tanh(M)\,X\cosh(MX)+\left[X-\frac{1}{\cosh^2(M)}\right]\sinh(MX)\right\}. \tag{22}$$

In a similar manner, as in the previous single filter scheme, one can obtain the integral transform solution for equations (21), and compose the final solution for the double filtering scheme as:

$$\theta(X,\tau)=e^{-b\tau}\,[\cosh(MX)-\tanh(M)\sinh(MX)]$$
$$-\frac{b}{2M}e^{-b\tau}\left\{-\tanh(M)\,X\cosh(MX)+\left[X-\frac{1}{\cosh^2(M)}\right]\sinh(MX)\right\}$$
$$+2\sum_{i=1}^{\infty}\frac{\sqrt{\mu_i^2-M^2}}{\mu_i^2-b}\left[\left(\frac{b}{\mu_i^2}\right)^2e^{-b\tau}-\frac{b}{\mu_i^2}e^{-\mu_i^2\tau}\right]\sin\left(\sqrt{\mu_i^2-M^2}\,X\right). \tag{23}$$

Additional filters can be superimposed on the present solution, at the cost of increasing analytical involvement, and there is a compromise associated with the relative gain in convergence rates that might result from such successive filtering, which can be better envisaged within the results section. Symbolic manipulation packages can be of particular interest in the realm of applications, once further filtering is decided on.

INTEGRAL BALANCE SCHEME

This acceleration technique is based on direct integration of the original partial differential equation over the whole domain and manipulation of the related boundary conditions and is fully described in references 5, 8 and 9. The aim is explicitly to extract and account for the equation and boundary source terms, again providing, to a certain extent, a convergence enhancement effect.

For the present application, equation. (3a) is now operated on with $\int_X^1 dX$, to yield, after recalling the inversion formula, equation (9b), and the ordinary differential system, equation (10a),

$$\frac{\partial\theta(X,\tau)}{\partial X}=\frac{\partial\theta(1,\tau)}{\partial X}-\sum_{i=1}^{\infty}\frac{1}{N_i}\left\{\int_X^1\psi_i(X)\,dX\right\}\left[g_i+(1-\mu_i^2)\bar{\theta}_i(\tau)\right] \tag{24}$$

Further integration of equation (24) with $\int_0^X dX$ and the use of equations (3c-d) yields

$$\theta(X,\tau)=\theta_b(\tau)-\sum_{i=1}^{\infty}\frac{1}{N_i}\left\{\int_0^X\int_X^1\psi_i(X)\,dX\right\}\left[g_i+(1-\mu_i^2)\bar{\theta}_i(\tau)\right] \tag{25}$$

and after evaluation of the above integral and substitution for the transformed potentials, equation (10c), one obtains,

$$\theta(X,\tau)\, t\, e^{-b\tau} + 2\sum_{i=1}^{\infty}\frac{\sqrt{\mu_i^2 - M^2}}{\mu_i^2 - b}\left[\frac{b - M^2}{\mu_i^2 - M^2}\, e^{-b\tau} - \frac{b}{\mu_i^2}\, e^{-\mu_i^2\tau}\right]\sin\left(\sqrt{\mu_i^2 - M^2}\, X\right). \tag{26}$$

Again, the boundary source term is explicitly accounted for, and the boundary conditions are identically satisfied, offering an expression with expected convergence improvement over the formal solution.

COMBINED FILTERING AND INTEGRAL BALANCE SCHEME

As an additional possibility to be investigated, the integral balance scheme will be employed right after the first filter application, i.e. on problem (15), in order to illustrate the relative merits of a combined procedure with the two schemes.

Following the same basic steps as in the previous section, one finds the final enhanced expression:

$$\theta(X,\tau) = \left(1 - \frac{b}{M^2}\right) e^{-b\tau}[\cosh(MX) - \tanh(M)\sinh(MX)] + \frac{b}{M^2}\, e^{-b\tau}$$

$$+ 2\sum_{i=1}^{\infty}\frac{\sqrt{\mu_i^2 - M^2}}{\mu_i^2 - b}\left[\frac{b}{\mu_i^2}\,\frac{b - M^2}{\mu_i^2 - M^2}\, e^{-b\tau} - \frac{b}{\mu_i^2}\, e^{-\mu_i^2\tau}\right]\sin\left(\sqrt{\mu_i^2 - M^2}\, X\right). \tag{27}$$

RESULT AND DISCUSSION

The four proposed improved expressions, equations (18), (23), (26) and (27), together with the plain formal solution, equation (12), are now critically compared against each other. As a reference solution, the splitting-up procedure[1-4], as described in the Appendix, offers an essentially ideal convergence behaviour for the present situation and a comparative pattern for the proposed solutions. It should be remembered, though, the severe limitations on the extension of the splitting-up approach to more involved problems.

As can be observed, solution (11a) may be interpreted as the sum of the homogeneous solution, as obtained through the separation of variables method, and a particular solution due to the non-homogeneities. Thus, the classical integral transform method may be regarded, roughly speaking, as an automatic technique to determine the particular solution of a partial differential equation. This particular solution, in a series form, does not have the desirable exponential convergence rate, and so, it may experience slow convergence. Therefore, all the convergence acceleration efforts are made in order to increment the rates of convergence for this series, and the schemes presented here are means of extracting partial sums of the series, since the splitting-up procedure, which allows for a closed form expression, is limited to very specific forms of time-dependent source functions. Another possible way, not investigated here, is to employ special techniques to directly enhance the summation itself, such as Aitken-Shanks transformations[12], Lanczos σ factors[13] or manipulations of the series[14].

In order to compare the different acceleration approaches, a computer program was developed to evaluate expressions (12), (18), (23), (26) and (27). It is of interest to study the influence, on the convergence rates, of the parameter M, the position X, the time τ and the decay constant b.

Figure 1 shows the convergence behaviour of the four acceleration approaches developed here against the formal solution, as a function of the parameter M. One can observe that the delay in convergence generally increases as M is also increased, especially for the integral balance approach. Except for this integral balance approach, the acceleration schemes present excellent performance through the range of M shown in *Figure 1*, the best technique being the double filtering. The worse behaviour of the integral balance can be explained through the fact that this approach makes explicit the boundary condition at $X = 0$ and, in the case of larger values of M, this condition has less effect on the temperatures in positions not so close to this boundary. Therefore, the integral balance is a convergence acceleration scheme to be recommended when the non-homogeneous boundary condition has a remarkable effect (as for $M < 1$, in the present situation) on the solution over the whole domain.

In practical applications, M is generally less than one and *Figure 1* shows that, in this case, "single filtering" is equivalent to "integral balance" while "double filtering" is equivalent to "combined filtering and integral balance". This can be observed from the two pairs of expressions, equations (18) and (26) and equations (23) and (27). Although formally different, the numerical values for each expression in the same pair above, are closer for $M < 1$. Then, only the filtering approaches will be considered in the analysis to follow.

Figure 2 shows the convergence process as the number of terms N in the series is increased. The effectiveness of the filtering techniques is quite noticeable from this plot. One may argue about the relative advantage brought up by the second filter, since the analytical development is more involved in this case. It is important to notice that the fin problem studied here can be used as a filter itself within another integral transform solution for a more complicated problem (e.g., a radiating fin, as an example of non-linear problem). In such circumstances, achieving convergence with somewhat fewer terms may result in a marked difference in computational costs.

Figure 3 brings the influence of the position within the medium, X, on the convergence behaviour. For the formal solution, as noted earlier, the boundary condition at $X = 0$ is not satisfied and this disturbs the convergence rates at the remaining positions. The behaviour of the

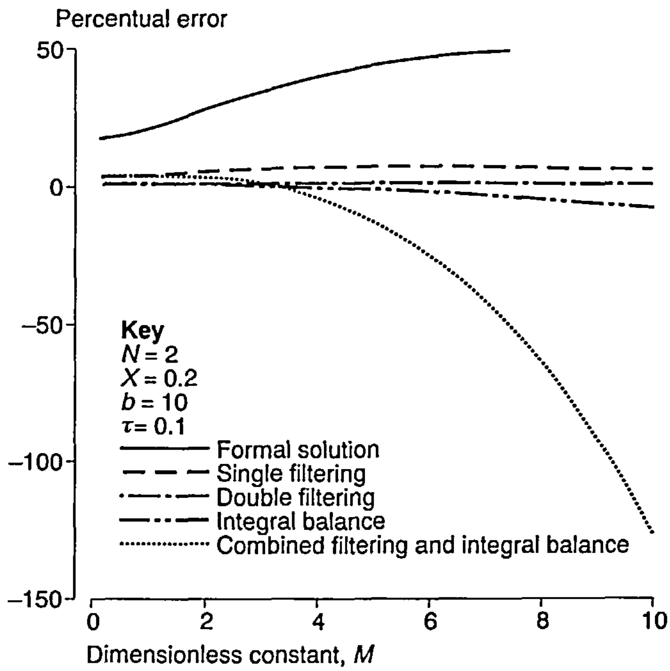

Figure 1 Influence of the parameter M on the convergence behaviour associated with the proposed solutions

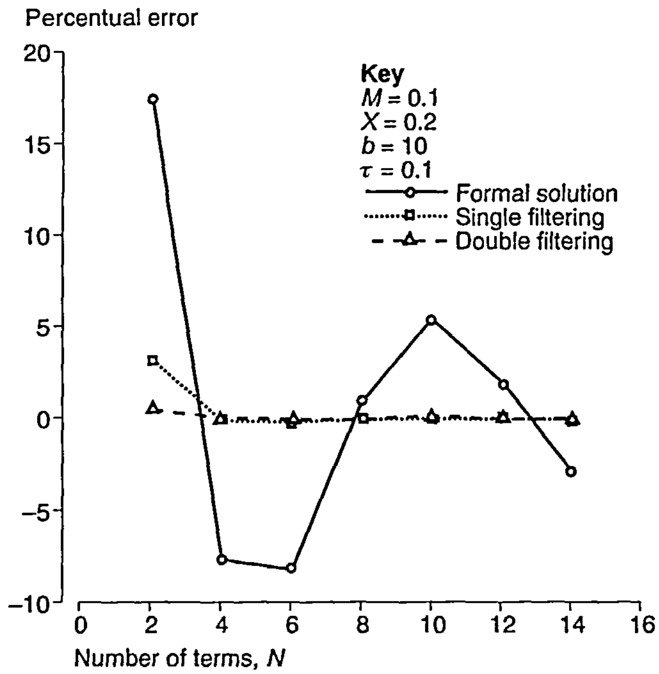

Figure 2 Convergence behaviour as a function of the number of terms considered in each series

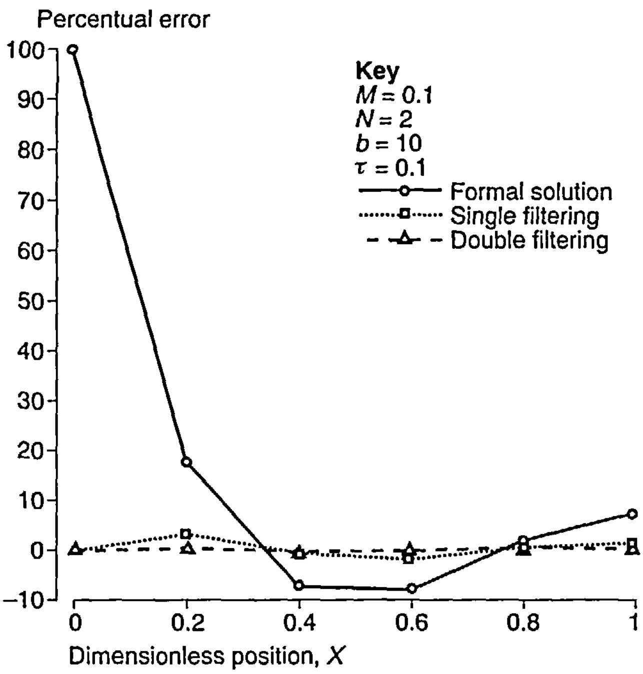

Figure 3 Influence of the position within the medium on the convergence behaviour

convergence with the time variable, τ, is shown in *Figure 4*, where, as expected, the convergence is improved as the time variable is increased. For the very small values of τ, it was in fact observed to be more efficient to filter once than twice. Finally, *Figure 5* shows the influence of the decay constant, b, on the convergence rates. As b increases, the convergence rates are all much closer, since, for a fixed time, a larger b means a boundary temperature approaching zero, which corresponds to the situation attained by the formal solution.

Table 1 presents numerical values for a representative case, including the splitting-up procedure for comparative purposes. This table demonstrates the anomalous convergence behaviour of the formal solution and the strong enhancement obtained with the various acceleration schemes, especially the splitting-up, that, unfortunately, is severely limited to specific functional forms of the related source terms.

Table 1 Numerical values for a comparative convergence behaviour: $M = 0.1$, $\tau = 0.1$ and $b = 10$

	Number of terms						
X	1	2	3	5	7	10	Exact
0.0	0.00000	0.00000	0 00000	0.00000	0.00000	0.00000	0.36788
	0.36788	0.36788	0.36788	0.36788	0.36788	0.36788	
	0.36788	0.36788	0.36788	0.36788	0.36788	0.36788	
	0.36788	0.36788	0.36788	0.36788	0.36788	0.36788	
	0.36788	0.36788	0.36788	0.36788	0.36788	0.36788	
	0.36788	0.36788	0.36788	0.36788	0.36788	0.36788	
0.2	0.35913	0.55828	0.66996	0.74590	0.70242	0.64031	0.67699
	0.58219	0.65508	0.67310	0.67882	0.67765	0.67668	
	0 65770	0.67376	0.67659	0.67703	0.67700	0.67699	
	0.58227	0.65510	0.67310	0.67882	0.67765	0.67668	
	0.65773	0.67376	0.67659	0.67703	0.67700	0.67699	
	0.70755	0.67709	0.67699	0.67699	0.67699	0.67699	
0.4	0.68311	0.91722	0.91722	0.81565	0.87665	0.83542	0.85531
	0.77561	0.86130	0.86130	0.85395	0.85566	0.85512	
	0.83694	0.85582	0.85582	0.85527	0.85532	0.85531	
	0.77567	0.86129	0.86129	0.85395	0.85566	0.85512	
	0.83697	0.85582	0.85582	0.85527	0.85532	0.85531	
	0.89112	0.85531	0.85531	0 85531	0.85531	0.85531	
0.6	0.94022	1.0163	0.90461	0.97147	0.92442	0.92741	0.94196
	0.92915	0.95699	0.93897	0.94306	0.94159	0.94181	
	0.93843	0.94456	0.94173	0.94200	0.94195	0.94196	
	0.92916	0.95698	0.93898	0.94305	0.94159	0.94181	
	0.93844	0.94456	0.94173	0.94200	0.94195	0.94196	
	0.95349	0.94185	0.94196	0.94196	0.94196	0.94196	
0.8	1.1053	0.96061	0.96061	0.95138	0.97158	0.96430	0.97670
	1.0277	0.97479	0.97479	0.97572	0.97661	0.97657	
	0.98816	0.97650	0.97650	0.97666	0.97669	0.97669	
	1.0277	0.97479	0.97479	0.97573	0.97661	0.97657	
	0.98814	0.97650	0.97650	0.97666	0.97669	0.97669	
	0.95456	0.97670	0.97670	0.97670	0.97670	0.97670	
1.0	1.1622	0.91602	1.0277	1.0095	1.0024	0.97361	0.98541
	1.0617	0.97162	0.98964	0.98635	0.98576	0.98530	
	1.0028	0.98298	0.98581	0.98545	0.98542	0.98541	
	1.0617	0.97163	0.98963	0.98635	0.98576	0.98530	
	1.0028	0.98298	0.98581	0.98545	0.98542	0.98541	
	0.94786	0.98552	0.98541	0.98541	0.98541	0.98541	

Note: The first line corresponds to the formal solution, equation (12), the second to single filtering, equation (18), the third to double filtering, equation (23), the fourth to integral balance, equation (26), the fifth to combined filtering and integral balance, equation (27), and the sixth to splitting-up (see Appendix).

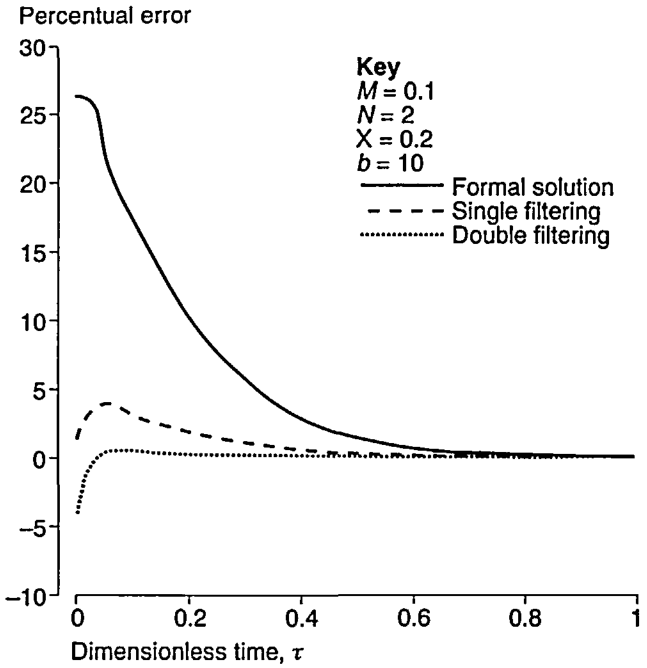

Figure 4 Influence of the time variable on the convergence behaviour

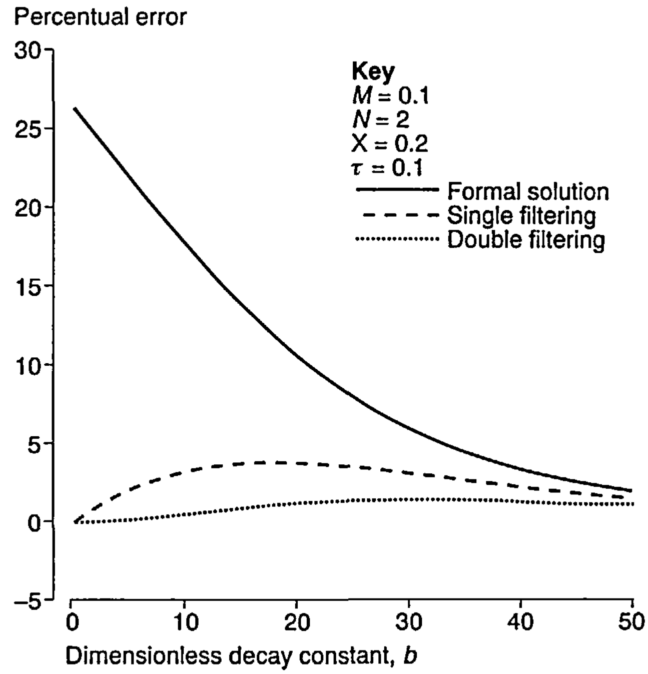

Figure 5 Influence of the decay constant b on the convergence behaviour

CONCLUSIONS

Many partial differential systems in the realm of applications have source terms formulations that disable the direct application of the well-known and effective splitting-up procedure, when applying the Integral Transform Method. On the other hand, the quite flexible filtering technique has presented an excellent performance, enhancing the convergence rates of the resulting eigenfunction expansions. Furthermore, this technique is not limited to linear problems that can be handled exactly by the classical Integral Transform Method, and may be used associated with the generalized approach (GITT) in the solution of more involved non-linear problems. Filters can be superimposed, yielding successive computational performance improvements, at the expense of increased analytical work. This may be particularly suitable for non-linear problems with considerable computational cost.

As shown by the present work, the integral balance approach is useful only when the non-homogeneous boundary condition represents a significant part of the final solution over the whole domain.

Although a general and universal filter is that one which incorporates, in full, the influence of the equation and boundary conditions source terms, with a reduced number of independent variables, practical considerations may lead us in proposing simplified filtering solutions, which only partially reproduce the information carried by the original source terms, but to a sufficient extent so as to enhance convergence rates and, at the same time, keep the overall analytical involvement to a reasonable level.

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APPENDIX: SPLITTING-UP PROCEDURE

The reader is referred to Reference 2 for the details on the splitting-up scheme. For the present case, problem (3) splits into

$$\theta(X,\tau) = \theta_{\phi}(X)\, e^{-b\tau} + \theta_t(X,\tau) \tag{A1}$$

where $\theta_{\phi}(X)$ is obtained from the following problem

$$\frac{d^2\theta_\phi}{dX^2} - (M^2 - b)\,\theta_\phi = 0, 0 < X < 1 \tag{A2a}$$

$$\theta_\phi(0) = 1 \tag{A2b}$$

$$\frac{d\theta_\phi(1)}{dX} = 0 \tag{A2c}$$

which furnishes

$$\theta_\phi(X) = \frac{1}{1 + e^{2\sqrt{M^2-b}}}\left[e^{\sqrt{M^2-b}(2-X)} + e^{\sqrt{M^2-b}X}\right], \text{ for } M^2 > b \tag{A3a}$$

or

$$\theta_\phi(X) = \tan(\sqrt{b - M^2})\sin(\sqrt{b - M^2}X) + \cos(\sqrt{b - M^2}X), \text{ for } M^2 < b. \tag{A3b}$$

Substituting equation (A1) into problem (3), one finds that $\theta_t(X,\tau)$ may be obtained from the following system

$$\frac{\partial\theta_t}{\partial\tau} = \frac{\partial^2\theta_t}{\partial X^2} - M^2\theta_t, 0 < X < 1, \tau > 0 \tag{A4a}$$

$$\theta_t(X, 0) = \theta_p(X) - \theta_\phi(X) \tag{A4b}$$

$$\theta_t(0, \tau) = 0 \tag{A4c}$$

$$\frac{\partial\theta_t(1, \tau)}{\partial X} = 0 \tag{A4d}$$

which is readily solved through the classical integral transform technique, to yield

$$\theta_t(X, \tau) = -2\sum_{i=1}^{\infty}\frac{\sqrt{\mu_i^2 - M^2}}{\mu_i^2 - b}\,\frac{b}{\mu_i^2}\,e^{-\mu_i^2\tau}\sin\left(\sqrt{\mu_i^2 - M^2}X\right). \tag{A5}$$

Then, the final solution is obtained, for each case ($M^2 < b$ or $M^2 > b$), by combining equations (A5) and equations (A3) as required by equation (A1). For $M^2 = b$, both solutions in equations (A3) can be used.

Comparison of equation (12) with equation (A5) shows that the splitting-up procedure gives the summation of the series related to the particular solution of the original partial differential equation in a closed form, providing the desirable exponential convergence rate, represented by the term $e^{-\mu_i^2 t}$.